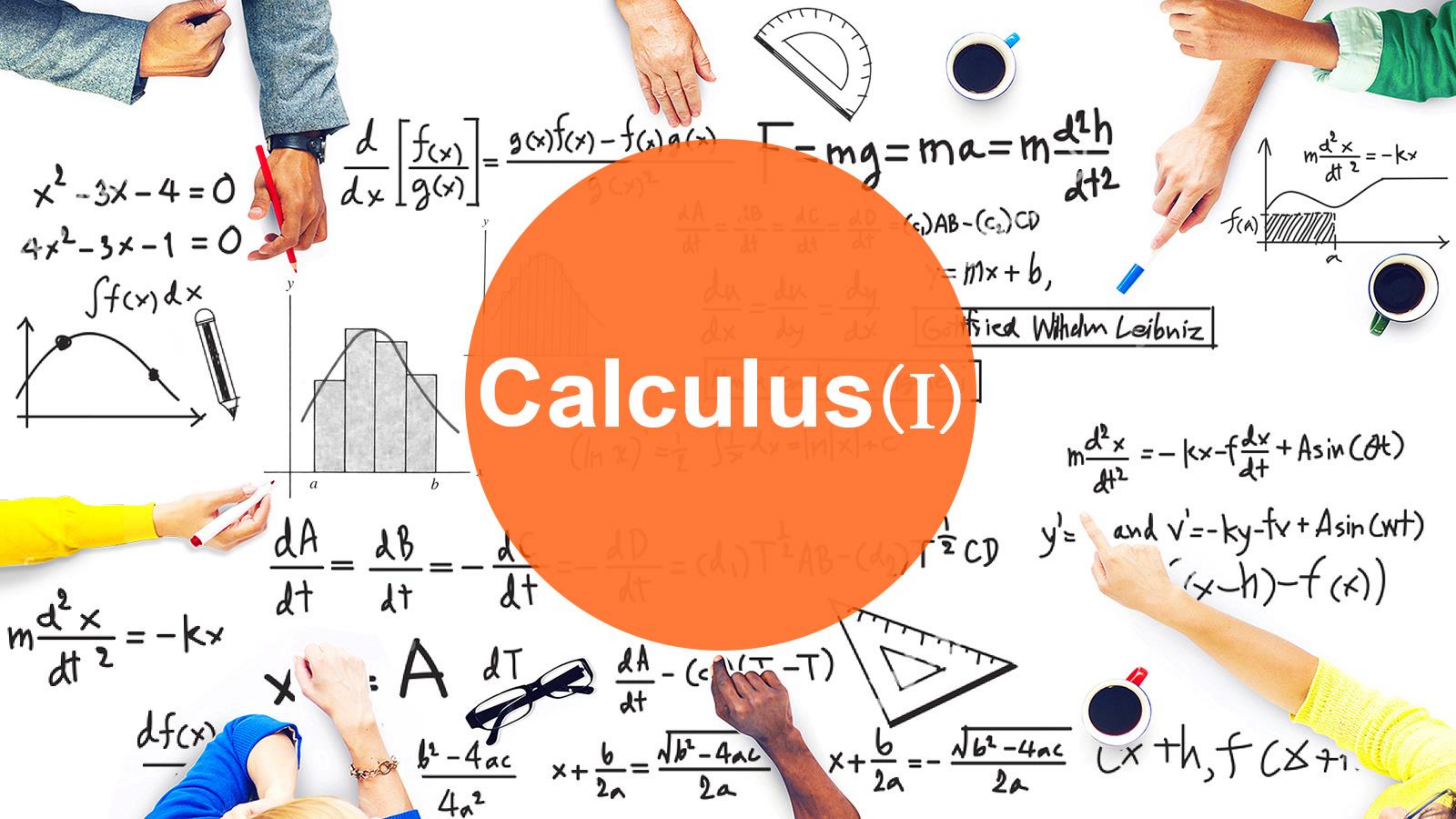
Calculus(Ⅰ)



Continuity of Functions

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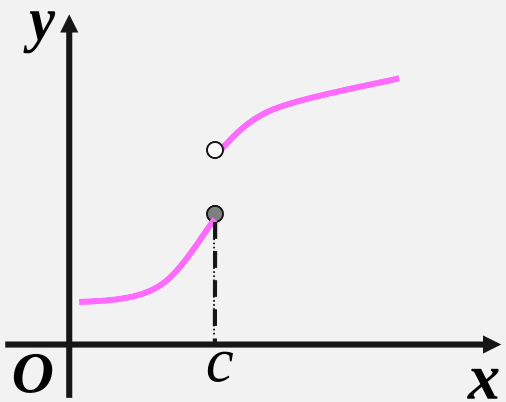
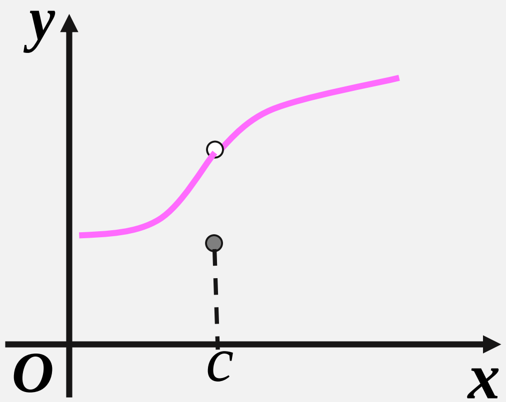
Problem Introduction

Which function is continuous at point c ?

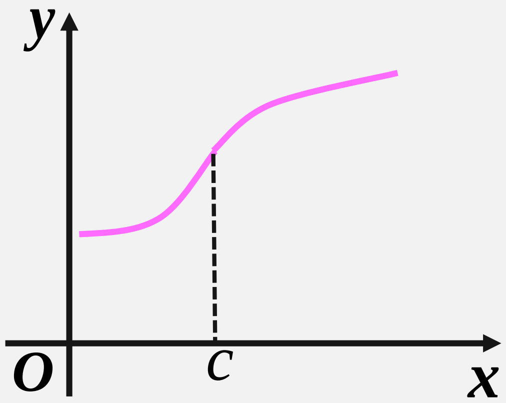


$\lim_{x \to c} f(x)$ does not exist.



$f(c)$ exists,but

$\lim_{x \to c} f(x) \neq f(c)$.



$\lim_{x \to c} f(x) = f(c)$.

Definition Continuity at a Point

Let f be defined on an open interval containing c, we say that f is continuous at c if

$$\lim_{x \to c} f(x) = f(c)$$

Require three things:

1. $\lim_{x \to c} f(x)$ exists,
2. $f(c)$ exists, (c is in the domain of f)
3. $\lim_{x \to c} f(x) = f(c)$.

Note: Any one of these three fails, then f is discontinuous at c.

Theorems of Continuity of Functions

Th A: **Continuity of Polynomial and Rational Functions**

- A polynomial function is continuous at every real number c.

- A rational function is continuous at every real number c in its domain.

(That is, everywhere except there its denominator is zero.)

Th B: **Continuity of Absolute Value and *n*th Root Functions**

- The absolute value function is continuous at every real number c.
- The nth root function is continuous at every real number c (n is odd)
- The nth root function is continuous at every positive real number c (n is even)

Theorems of Continuity of Functions

Th C: **Continuity under Function Operations**

If f, g are continuous at c, then the following expresssions are continuous.

$$kf,\ f \pm g,\ f \cdot g,\ f/g\,(g(c) \neq 0),\ f^{n},\ f^{\frac{1}{n}}\ \left(f(c) > 0, n \text{ is even}\right).$$

Theorems of Continuity of Functions

Th D: **Continuity of Trigonometric Functions**

$\sin x, \cos x, \tan x, \cot x, \sec x, \csc x$ are continuous at every real number c in their domains.

Th E: **Composite Limit Theorem**

$$\lim_{x\to c} g(x) = L,\ f \text{ is continuous at } L$$

$$\Rightarrow \lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right) = f(L).$$

Definition Continuity on an Interval

1. f is continuous in (a,b)
2. f is right continuous at a: $\lim_{x\to a^+} f(x) = f(a)$
3. f is left continuous at b: $\lim_{x\to b^-} f(x) = f(\mathrm{b})$

$\Bigg\}$ f is continuous on $[a,b]$

Theorems of Continuity of Functions

Th F: **Intermediate Value Theorem**

$$\left.\begin{array}{l}\text{If } f \text{ is defined on } [a,b] \\ f \text{ is continuous on } [a,b] \\ \forall w \in [f(a), f(b)]\end{array}\right\} \Rightarrow \exists c \in [a,b], \text{such that } f(c) = w.$$

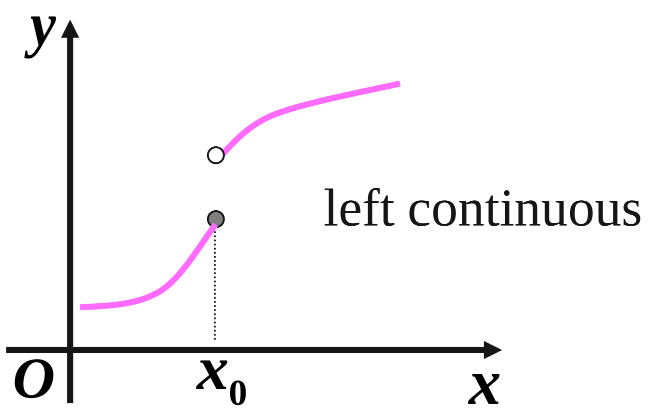
Definitions of Continuity of Functions

Left Continuity

$$\lim_{x \to x_0^-} f(x) = f(x_0) \quad \left(f(x_0^-) = f(x_0) \right)$$

$f(x)$ is left continuous at x_0

(continuity from the left)

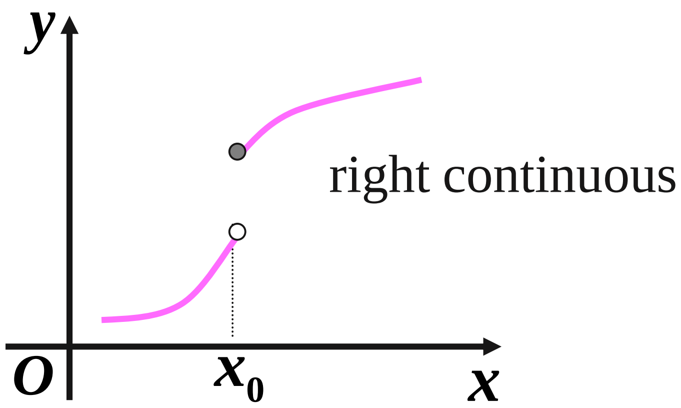


Right Continuity

$$\lim_{x \to x_0^+} f(x) = f(x_0) \quad \left(f(x_0^+) = f(x_0) \right)$$

$f(x)$ is right continuous at x_0

(continuity from the right)



$f(x)$ is continuous at $x = x_0 \Leftrightarrow f(x)$ is left and right continuous at $x = x_0$.

Example 1

By definition $\lim_{x\to c} f(x) = f(c)$

Prove that $f(x) = \begin{cases} x\sin\dfrac{1}{x}, & x \neq 0, \\ 0, & x = 0, \end{cases}$ is continuous at $x = 0$.

$$\because \lim_{x\to 0} x\sin\frac{1}{x} = 0,$$

and $f(0) = 0$,

$$\lim_{x\to 0} f(x) = f(0),$$

$\therefore$ $f(x)$ is continuous at $x = 0$.

Example 2

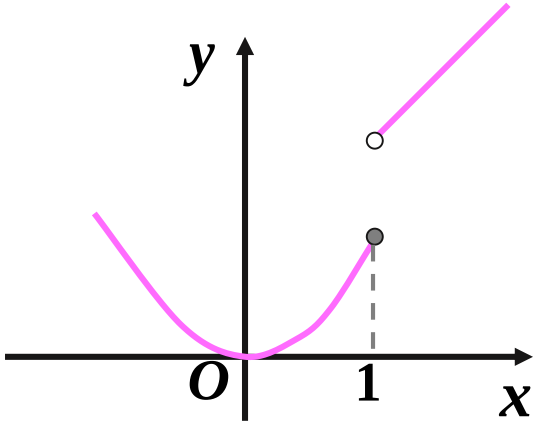
Determine $f(x)=\begin{cases} x^2, & x \le 1, \\ x+1, & x>1, \end{cases}$ is continuous at $x=1$ or not ?

$\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} x^2 = 1 = f(1),$

$\therefore$ $f(x)$ is left continuous at $x=1$,

$\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+}(x+1) = 2 \ne f(1),$

but is not right continuous at $x=1$,



$\therefore$ $f(x)$ is not continuous at $x=1$.

Summary of Continuity of Functions

Definition: f is continuous at $c \Leftrightarrow \lim_{x \to c} f(x) = f(c)$

Require three things:

1. $\lim_{x \to c} f(x)$ exists
2. $f(c)$ exists
3. $\lim_{x \to c} f(x) = f(c)$

$f(x)$ is continuous at $x = x_0 \Leftrightarrow f(x)$ is left and right continuous at $x = x_0$.

Questions and Answers

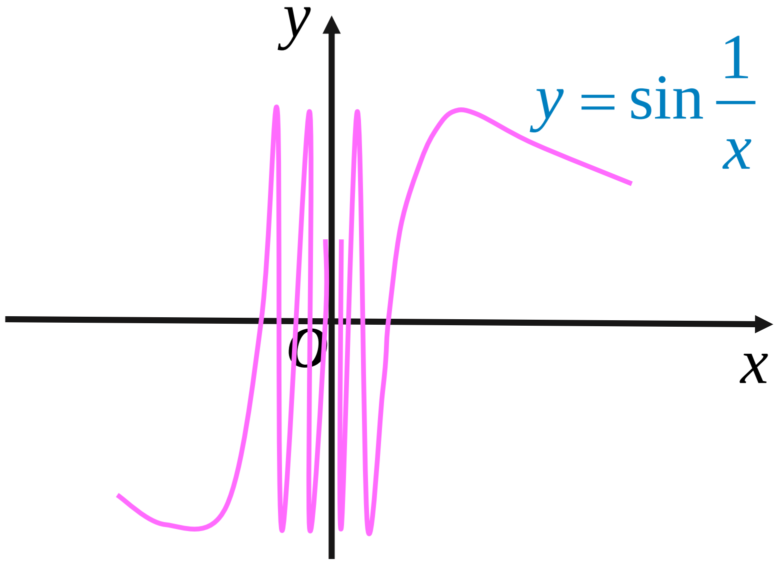
Q1: $f(x)=\begin{cases}\sin\frac{1}{x}, & x\neq 0,\\ 0, & x=0,\end{cases}$ is continuous at $x=0$ or not?

Although $f(0)=0$,

but $\lim\limits_{x\to 0} f(x)$ does not exist.

$\left(x\to 0,\ f(x)=\sin\frac{1}{x}\ \text{can jump between}\ -1\ \text{and}\ 1\right)$

$\therefore\ f(x)$ is not continuous at $x=0$.



Questions and Answers

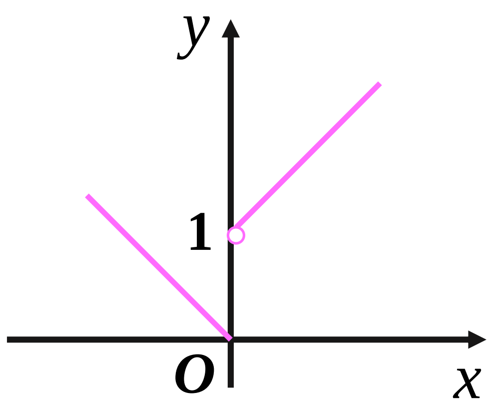
Q2: $f(x)=\begin{cases} -x, & x\le 0, \\ 1+x, & x>0, \end{cases}$ is continuous at $x=0$ or not?

Although $f(0)=0$

$\lim\limits_{x\to 0^-}(-x)=0, \quad \lim\limits_{x\to 0^+}(1+x)=1,$

$f(0^-)\neq f(0^+) \Rightarrow \lim\limits_{x\to 0} f(x)$ does not exist

$\therefore\ f(x)$ is not continuous at $x=0$.




Continuity